Combinatorial Networks, 2015 Spring
Homework 2

1. Prove that for any integer $k > 0$, there exists an integer $n := n(k)$ such that the following holds. Any 2-edge coloring of any tournament on n vertices has a monochromatic directed path on k vertices.

2. Prove the Richardson's Theorem that any directed bipartite planar graph has a kernal.

3. Let k be even. Prove that the edge set of the complete graph K_k (with k vertices) can be partitioned into $(k-1)$ perfect matchings. That is, $\chi'(K_k) = k-1$.

4. Let $K_{n,n}$ be the complete bipartite graph on $n = \binom{2k-1}{k}$ vertices. Think of every vertex v in each partition class of $K_{n,n}$ as representing a subset $S_v \subset [2k-1]$ of size k, and assign v the list of colors S_v. Show that there is no legal coloring of its vertices from the lists.

5. Suppose X is a set of n elements, and $S_1, ..., S_m$ are m subsets of X of average size at least n/w. Show that if $m \geq 2kw^k$, then there are k distinct sets $S_{i_1}, ..., S_{i_k}$ satisfying $|S_{i_1} \cap ... \cap S_{i_k}| \geq n/2w^k$.

6. Suppose there are m red clubs $R_1, ..., R_m \subseteq [n]$ and m blue clubs $B_1, ..., B_m \subseteq [n]$ such that $|R_i \cap B_i|$ is odd for every i, and $|R_i \cap B_j|$ is even for every $i \neq j$. Show that $m \leq n$.

Can you weaken the second condition to $i < j$?

7. Suppose $R_1, ..., R_m \subseteq [n]$ is a club satisfying that $|R_i| \neq 0 \mod 6$ for every i, and $|R_i \cap R_j| = 0 \mod 6$ for every $i \neq j$. Prove that $m \leq 2n$.